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Symmetries, Conservation Laws and Multipliers via Partial Lagrangians and Noether's Theorem for Classically Non-Variational Problems

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Abstract We show how one can construct conservation laws of equations that are not variational but are Euler–Lagrange in *part* using Noether-type symmetries associated with partial Lagrangians. These Noether-type symmetries are, usually, not symmetries of the system. The resultant construction of the conservation law resorts to a formula equivalent to Noether's theorem. A variety of examples are given.

1 Introduction

A systematic and, by now a well known, way of determining conservation laws for systems of Euler–Lagrange equations once their Noether symmetries are known is via Noether theorem [7, 8]. This theorem relies on the availability of a Lagrangian and the corresponding Noether symmetries which leave invariant the action integral. There are large classes of differential equations that do not admit a Lagrangian, e.g. scalar evolution equations (see e.g. [3]). For these equations, there are approaches that do not assume the existence of a Lagrangian. The most elementary of these is the direct method which has been extensively used for the construction of conserved quantities for well-known differential equations. Another method involves writing the conservation law in characteristic form [8]. Anco and Bluman [1] described a detailed way of using a knowledge of the characteristics (multipliers) to construct the conservation laws—once a multiplier is known, a formula is used

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to construct the conserved density or flux. In the case of Euler–Lagrange equations, these multipliers are the (characteristics of the corresponding) Noether symmetries—the relationship between symmetries and multipliers in a general situation have only recently been established by Anco and Kara [2]. Furthermore, computer algebra programs have been developed for the approaches which do not rely on a Lagrangian (see [10]). Relationships between symmetries and conservation laws beyond Noether's theorem have been done by Kara and Mahomed [4, 5], Anco and Bluman [1] and Vinogradov [9] inter alia. Despite the existence of all the alternative methods, one would still look to a formula type method to determine conservation laws especially for the non variational case.

In [6], a new method for constructing conservation laws via operators that are not necessarily symmetries of the underlying system was presented. These *Noether-type* symmetries which are associated with *partial* Lagrangians aid, via a formula, in the construction of conservation laws of the underlying system which need not have Lagrangians. It turns out that the multipliers which, as indicated above are symmetries only in the Euler–Lagrange equations, are the Noether-type symmetries following the partial Lagrangian in the general situation. Moreover, the relationship between the multipliers and the Lie symmetries of the equation are satisfied as described in [2]. For a deeper, theoretical aspects of this study, for e.g., the theorems regarding the resultant algebraic properties that arise, we refer the reader to [6].

The examples we consider below are not usually analysed using Lagrangian methods. We describe the construction of their conservation laws via partial Lagrangians. We include a system of p.d.e.s as well as fourth-order p.d.e.s. Firstly, we present the notations and notions of Noether-type symmetries and partial Lagrangians.

Suppose (t, x) and (u, v) are the independent and dependent variables, respectively. The *total derivative operator* with respect to t is

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x} + \cdots$$

and equivalently for D_x. The Euler–Lagrange operators are

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t^2 \frac{\partial}{\partial u_{tt}} + D_x^2 \frac{\partial}{\partial u_{xx}} + \cdots, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t \frac{\partial}{\partial v_t} - D_x \frac{\partial}{\partial v_x} + D_t^2 \frac{\partial}{\partial v_{tt}} + D_x^2 \frac{\partial}{\partial v_{xx}} + \cdots. \end{aligned} \tag{1}$$

A *Lie symmetry generator* will be denoted by

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial u} + \zeta \frac{\partial}{\partial v} \tag{2}$$

where, for a point generator, ξ, τ, ϕ and ζ are functions of (t, x, u, v).

Consider a k-th order differential equation

$$\begin{aligned} E^1(t, x, u, v, u_t, u_x, v_t, v_x, \ldots) = 0, \\ E^2(t, x, u, v, u_t, u_x, v_t, v_x, \ldots) = 0. \end{aligned} \tag{3}$$

A *conserved flow* of (3) (T^1, T^2) is a vector along which the conservation law

$$D_t T^1 + D_x T^2 = 0 \tag{4}$$

is satisfied along the solutions of (3). If there exists a function $L(t,x,u,v,u_t,u_x,v_t,v_x,\ldots)$ such that

$$\frac{\delta L}{\delta u}=0,\qquad \frac{\delta L}{\delta v}=0 \tag{5}$$

satisfies (3), we say (3) is variational and L is a Lagrangian of (3). If (5) does not satisfy (3) completely but

$$\frac{\delta L}{\delta u}=E_0^1,\qquad \frac{\delta L}{\delta v}=E_0^2 \tag{6}$$

where $E^1=E_0^1+\cdots$ and $E^2=E_0^2+\cdots$, we say L is a *partial Lagrangian* of (3). A generator of the type X in (2) is a *Noether type symmetry* corresponding to a partial Lagrangian L if it satisfies

$$XL+L(D_t\tau+D_x\xi)=W^1\frac{\delta L}{\delta u}+W^2\frac{\delta L}{\delta v}+D_tf+D_xg \tag{7}$$

for some gauge vector (f,g), $W^1=\phi-u_t\tau-u_x\xi$, $W^2=\zeta-v_t\tau-v_x\xi$ and X prolonged accordingly.

Note: For Euler–Lagrange equations, (7) reduces to $XL+L(D_t\tau+D_x\xi)=D_tf+D_xg$ and X is a Noether symmetry which leaves invariant the action integral and is also a Lie symmetry of the Euler–Lagrange equations. For partial Lagrangians, the Noether type symmetries need not be symmetries of the differential equation (see [6]).

Corresponding to each Noether type symmetry X of partial Lagrangian L of first-order, there exists a conserved flow (T^1,T^2) of the system (3) given by

$$\begin{aligned} T^1&=L\tau+W^1\frac{\partial L}{\partial u_t}+W^2\frac{\partial L}{\partial v_t}-f,\\ T^2&=L\xi+W^1\frac{\partial L}{\partial u_x}+W^2\frac{\partial L}{\partial v_x}-g. \end{aligned} \tag{8}$$

For second-order Lagrangians, (8) becomes

$$\begin{aligned} T^1&=L\tau+W^1\frac{\partial L}{\partial u_t}+W^2\frac{\partial L}{\partial v_t}+[D_jW^\alpha-W^\alpha D_j]\frac{\partial L}{\partial u^\alpha_{tj}}-f,\\ T^2&=L\xi+W^1\frac{\partial L}{\partial u_x}+W^2\frac{\partial L}{\partial v_x}+[D_jW^\alpha-W^\alpha D_j]\frac{\partial L}{\partial u^\alpha_{xj}}-g, \end{aligned} \tag{9}$$

where $u^1=u$ and $u^2=v$. These ‘formulae’ for the conserved flow are the same as those in Noether’s theorem even though the generator X is not a Lie symmetry of (3).

2 Applications

We consider a range of examples of differential equations that either do not admit the usual Lagrangians or finding one is not trivial. These range from examples in the literature that include systems of second-order p.d.e.s (Schrödinger) to a fourth-order p.d.e. (Boussinesq).

Example 1—Illustrative (The modified Burgers equation) Consider the equation

$$u_t+u^2u_x+u_{xx}=0. \tag{10}$$

If we define a partial Lagrangian $L = u_x^2/2$, $\delta L/\delta u = -u_{xx}$ so that (10) can be written as $u_t + u^2 u_x = -\delta L/\delta u$ and, therefore, $\delta L/\delta u$ can be replaced by $-u_t - u^2 u_x$ to determine the Noether-type generators, by (7), $X = \tau\partial/\partial t + \xi\partial/\partial x + \eta\partial/\partial u$ corresponding to L. That is,

$$\phi^x \frac{\partial L}{\partial u_x} + \frac{1}{2}u_x^2(\tau_t + u_t\tau_u + \xi_x + u_x\xi_u)$$
$$= (\phi - u_t\tau - u_x\xi)(u_t + u^2 u_x) + f_t + u_t f_u + g_x + u_x g_u \tag{11}$$

where f and g are functions of (x, t, u) and, after expansion of the prolongation ϕ^x, separation by the derivative terms lead to

$$\begin{aligned}
u_x^3&: \quad \xi_u = 0,\\
u_x^2 u_t&: \quad \tau_u = 0,\\
u_x^2&: \quad \phi_u = 0,\\
u_t^2&: \quad \tau = 0,\\
u_x u_t&: \quad \xi = 0,\\
u_x&: \quad g = -\frac{1}{3}u^3\phi + \phi_x u + \alpha(x,t),\\
u_t&: \quad f = -u\phi + \beta(x,t),\\
&: \quad f_t + g_x = 0.
\end{aligned} \tag{12}$$

Substitution into the last equation in (12) and separating by powers of u, we get $\phi_t = \phi_x = 0$, $\alpha_x + \beta_t = 0$ so that $T^1 = -u + \beta$ and $T^2 = -u_x - \frac{1}{3}u^3 + \alpha$ which is the standard conserved flow of (10).

Remark 1 We were able to find the conserved flow by resorting Noether like symmetries through a part of the equation being of Euler–Lagrange form. Here, we have not obtained the association of symmetries and conservation laws as discussed in the sense of Kara and Mahomed [4]. However, as will be more evident in the subsequent examples, the multipliers are always of the form W, i.e., $D_t T^1 + D_x T^2 = W(u_t + u^2 u_x + u_{xx})$ and W is directly related to the Lie symmetries of the equation [2].

Example 2 (A sigma model equation) The general sigma model equation is of the form

$$u_{xt} = (u_x u_t)^p. \tag{13}$$

The case $p = 1$ is the well known one and of greatest interest. We show how a complete class of Noether type conserved flows may be constructed using a partial Lagrangian $L = \frac{1}{2}u_x u_t$. The determining equations for the Noether type generators $X = \tau\partial/\partial t + \xi\partial/\partial x + \eta\partial/\partial u$ are given by

$$\frac{1}{2}u_x\phi^t + \frac{1}{2}u_t x\phi^x + \frac{1}{2}u_t u_x(\tau_t + u_t\tau_u + \xi_x + u_x\xi_u)$$
$$= -(\phi - u_t\tau - u_x\xi)(u_t u_x) + f_t + u_t f_u + g_x + u_x g_u. \tag{14}$$

The relevant separation by derivative terms lead to

$$
\begin{aligned}
&u_x^2u_t\colon && \xi=-\frac{1}{2}\xi_u \quad \text{so that} \quad \xi=b(x,t)e^{-2u},\\
&u_t^2u_x\colon && \tau=-\frac{1}{2}\tau_u \quad \text{so that} \quad \tau=a(x,t)e^{-2u},\\
&u_x^2\colon && \xi_t=0,\\
&u_t^2\colon && \tau_x=0,\\
&u_xu_t\colon && -\phi=\phi_u \quad \text{so that} \quad \phi=c(x,t)e^{-u},\\
&u_t\colon && \frac{1}{2}\phi_x=f_u \quad \text{so that} \quad f=-\frac{1}{2}c_xe^{-u}+d(x,t),\\
&u_x\colon && \frac{1}{2}\phi_t=g_u \quad \text{so that} \quad g=-\frac{1}{2}c_te^{-u}+e(x,t),\\
&\colon && f_t+g_x=0 \quad \text{so that} \quad c_{xt}=0 \quad \text{and} \quad d_t+e_x=0.
\end{aligned}
\tag{15}
$$

It is clear that $a=a(t)$ and $b=b(x)$. Thus, by an adaptation of (8), the conserved density and flux are infinite and are, respectively,

$$
\begin{aligned}
T^1&=\frac{1}{2}au_xu_te^{-2u}+\frac{1}{2}u_x(ce^{-u}-au_te^{-2u}-bu_xe^{-2u})-\left(-\frac{1}{2}c_xe^{-u}+d(x,t)\right),\\
T^2&=\frac{1}{2}bu_xu_te^{-2u}+\frac{1}{2}u_t(ce^{-u}-au_te^{-2u}-bu_xe^{-2u})-\left(-\frac{1}{2}c_te^{-u}+d(x,t)\right).
\end{aligned}
\tag{16}
$$

It can be shown that the conservation law is $D_tT^1+D_xT^2=(ce^{-u}-au_te^{-2u}-bu_xe^{-2u})(u_{xt}-u_xu_t)$. Note that the multiplier $W=ce^{-u}-au_te^{-2u}-bu_xe^{-2u}$ appears as in the case of the usual Noether's theorem corresponding to the Noether type symmetry. Also, the factor (or the Noether type symmetry X) has no relationship to the symmetries of the equation.

As an example, let $c=x-t$ and $a=b=0$, $T^1=\frac{1}{2}(x-t)e^{-u}u_x+\frac{1}{2}e^{-u}$ and $T^2=\frac{1}{2}(x-t)e^{-u}u_t-\frac{1}{2}e^{-u}$ which provides nontrivial conserved vector of the p.d.e.

Example 3 (A system of p.d.e.s) The Schrödinger equation

$$iw_t+w_{xx}+F(|w|)w=0 \tag{17}$$

written as a system of second-order p.d.e.s, by letting $w=u+iv$, is

$$u_t=-v_{xx}-F(u^2+v^2)v, \qquad v_t=F(u^2+v^2)u+u_{xx}. \tag{18}$$

A partial Lagrangian is $L=\frac{1}{2}u_x^2+\frac{1}{2}v_x^2$ so that $\delta L/\delta u=-u_{xx}=Fu-v_t$ and $\delta L/\delta v=-v_{xx}=Fv+v_t$. The defining equation for the Noether type symmetries (7) becomes

$$
\begin{aligned}
&\phi^xu_x+\zeta^xv_x+\left(\frac{1}{2}u_x^2+\frac{1}{2}v_x^2\right)(\tau_t+u_t\tau_u+v_t\tau_v+\xi_x+u_x\xi_u+v_x\xi_v)\\
&\quad=(\phi-u_t\tau-u_x\xi)(Fu-v_t)+(\zeta-v_t\tau-v_x\xi)(Fv+u_t)\\
&\qquad+f_t+u_tf_u+v_tf_v+g_x+u_xg_u+v_xg_v
\end{aligned}
\tag{19}
$$

in which X is of the form in (2). The third-order derivative monomials show that $\tau=\tau(x,t)$ and $\xi=\xi(x,t)$. The second-order monomials are

$$\begin{aligned}
&u_x^2: && -\xi_x+\phi_u+\frac{1}{2}\tau_t=0,\\
&v_x^2: && -\xi_x+\zeta_v+\frac{1}{2}\tau_t=0,\\
&u_xv_x: && \phi_v+\zeta_u=0,\\
&u_tu_x: && \tau_x=0,\\
&u_xv_t: && \xi=0,
\end{aligned}\tag{20}$$

so that $\phi=-\frac{1}{2}\tau_t u+\alpha(x,t,v)$ and $\zeta=-\frac{1}{2}\tau_t v+\beta(x,t,u)$ where $\alpha_v+\beta_u=0$. Thus, $\phi=-\frac{1}{2}\tau_t u+M(x,t)v+N(x,t)$ and $\zeta=-\frac{1}{2}\tau_t v+P(x,t)u+Q(x,t)$. The remaining separations are

$$\begin{aligned}
&u_x: && g_u=\phi_x,\\
&v_x: && g_v=\zeta_x,\\
&u_t: && f_u=Fu\tau-\zeta,\\
&v_t: && f_v=Fv\tau+\phi,\\
&: && \phi Fu+\zeta Fv+f_t+g_x=0.
\end{aligned}\tag{21}$$

As a sample case, we suppose $F=\sqrt{u^2+v^2}$. The first three equations lead to $f=\frac{1}{3}(u^2+v^2)^{\frac{3}{2}}K-\frac{1}{2}Pu^2-Qu+\frac{1}{2}(P+\mu(t))v^2+Nv+S(x,t)$, $g=(P_x+N_x)u+Q_xv+R(x,t)$, $\tau=K$, $\phi=(P+\mu)v+N$ and $\zeta=Pu+Q$, where K is a constant. Substituting into the last equation in (21), the separation leads to

$$\begin{aligned}
&uv\sqrt{u^2+v^2}: && \mu+2P=0,\\
&u\sqrt{u^2+v^2}: && N=0,\\
&v\sqrt{u^2+v^2}: && Q=0,\\
&u^2: && P_t=0,\\
&v^2: && P_t+\mu_t=0,\\
&uv: && P_{xx}=0,\\
&: && S_t+R_x=0
\end{aligned}\tag{22}$$

so that $\mu=K_1$, a constant, and two cases arise:

(i) $K=1$, $K_1=0$ yields the Noether type symmetry $X=\partial/\partial t$, $f=\frac{1}{3}(u^2+v^2)^{\frac{3}{2}}$ and $g=0$. The conserved flow is given by

$$T^1=\frac{1}{2}(u_x^2+v_x^2)-\frac{1}{3}(u^2+v^2)^{\frac{3}{2}},\qquad T^2=-u_xu_t-v_xv_t.\tag{23}$$

The conservation law is

$$D_tT^1+D_xT^2=-u_t(-v_t+\sqrt{u^2+v^2}u+u_{xx})-v_t(u_t+v_{xx}+\sqrt{u^2+v^2}v).$$

General note. The multipliers are in fact $W^1=-u_t$ and $W^2=-v_t$ which are unrelated, as factors in the sense of Noether's theorem, to the symmetries of the equation. Generally, these multipliers are symmetries only in the variational case. Here, these multipliers are the Noether type symmetries in this the partial variational case—see the works of Anco and Bluman [1].

(ii) $K_1 = 1$, $K = 0$ leads to the Noether type symmetry $X = \frac{1}{2}v\partial_u - \frac{1}{2}u\partial_v$ conserved flow

$$T^1 = -\left(\frac{1}{4}u^2 + \frac{1}{4}v^2\right), \qquad T^2 = \frac{1}{2}vu_x - \frac{1}{2}uv_x. \tag{24}$$

Here, conservation laws is

$$D_tT^1 + D_xT^2 = \frac{1}{2}v(-v_t + \sqrt{u^2+v^2}u + u_{xx}) - \frac{1}{2}u(u_t + v_{xx} + \sqrt{u^2+v^2}v).$$

However, we needed to add and subtract the term $uv\sqrt{u^2+v^2}$. Again, the general comment regarding the multipliers being the Noether type symmetries apply.

Example 4 (Boussinesq equation) The are various versions of the Boussinesq equation which models the behavior of long waves in a shallow medium either written as a second-order system or the fourth-order

$$u_{xxxx} + uu_{xx} + u_x^2 + u_{tt} = 0 \tag{25}$$

whose Lie algebra of point symmetry generators are

$$Y_1 = x\partial_x + 2t\partial_t - 2u\partial_u, \qquad Y_2 = \partial_x, \qquad Y_3 = \partial_t. \tag{26}$$

The Noether type symmetries via the partial Lagrangian $L = \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_t^2 - \frac{1}{2}uu_x^2$ will be determined using (7)—here, $\frac{\delta L}{\delta u} = -\frac{1}{2}u_x^2$. In this case, XL is a second prolongation of X so that

$$\begin{aligned} XL = {} & -\frac{1}{2}\phi u_x^2 + u_tu_x\xi_t + u_t^2u_x\xi_u + uu_x^3\xi_u + uu_x^2\xi_x \\ & + u_t^2\tau_t + u_t^3\tau_u + uu_tu_x^2\tau_u + uu_tu_x\tau_x - u_t\phi_t \\ & - u_t^2\phi_u - uu_x^2\phi_u - uu_x\phi_x - 2u_x\tau_uu_{x,t}u_{x,x} \\ & - 2\tau_xu_{x,t}u_{x,x} - 3u_x\xi_uu_{x,x}^2 - 2\xi_xu_{x,x}^2 \\ & - u_t\tau_uu_{x,x}^2 + \phi_uu_{x,x}^2 - u_x^3u_{x,x}\xi_{u,u} - 2u_x^2u_{x,x}\xi_{x,u} \\ & - u_xu_{x,x}\xi_{x,x} - u_tu_x^2u_{x,x}\tau_{u,u} - 2u_tu_xu_{x,x}\tau_{x,u} \\ & - u_tu_{x,x}\tau_{x,x} + u_x^2u_{x,x}\phi_{u,u} + 2u_xu_{x,x}\phi_{x,u} \\ & + u_{x,x}\phi_{x,x} \end{aligned} \tag{27}$$

which when substituted into (7) and going through the usual separation leads to

$$\begin{aligned} & \tau = 0, \qquad \xi = 0, \qquad \phi = A + Bt + Cx + Dxt, \\ & f = -(B + Dx)u + a(x,t), \qquad g = -\frac{1}{2}(C + Dt)u^2 + b(x,t) \end{aligned} \tag{28}$$

where $a_t + b_x = 0$ and A, B, C and D are arbitrary constants leading to four Noether type symmetries

$$\partial_u, \qquad t\partial_u, \qquad x\partial_u, \qquad xt\partial_u. \tag{29}$$

None of these is a symmetry of (25)—the multipliers are, respectively,

$$W_0 = 1, \qquad W_1 = t, \qquad W_2 = x, \qquad W_3 = xt \tag{30}$$

The relationship between these multipliers and the Lie symmetries in (26) is

$$\begin{aligned} &Y_i Q_0 = 0, \quad i = 1, \dots, 3, \\ &Y_1 Q_1 = 2Q_1, \qquad Y_2 Q_1 = 0, \qquad Y_3 Q_1 = 1, \\ &Y_1 Q_2 = Q_2, \qquad Y_2 Q_2 = 1, \qquad Y_3 Q_2 = 0, \\ &Y_1 Q_3 = 3Q_3, \qquad Y_2 Q_3 = Q_1, \qquad Y_3 Q_3 = Q_2. \end{aligned} \tag{31}$$

The significance and general theorems of this result is the subject of Anco and Kara [2]. We, thus, have four nontrivial conservation laws given by the formula in (9) for one dependent variable u.

For example, $D = 1$ (Noether type symmetry $xt\partial_u$) yields the conserved density and flux

$$\begin{aligned} T^1 &= -xtu_t, \\ T^2 &= -xtuu_x + tu_{xx} - xtu_{xxx} + \frac{1}{2}tu^2 \end{aligned} \tag{32}$$

so that $D_t T^1 + D_x T^2 = -xt(u_{xxxx} + uu_{xx} + u_x^2 + u_{tt})$.

3 Concluding Remarks

We have shown that for a system of partial differential equations which admit partial Lagrangians how one can construct conservation laws via Noether-type symmetries using a formula equivalent to Noether's theorem. These Noether-type symmetries are in general not symmetries of the underlying equations. Moreover, the Noether-type symmetries do not form a Lie algebra. In this paper the approach developed was in general for multi-variables. We made various applications to partial differential equations.

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